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LETTER TO THE EDITOR

On the Fokker–Planck equation with force term

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Abstract. We show that half-range completeness and half-range orthogonality relations hold for the stationary one-dimensional Fokker-Planck transport problem for some velocity-dependent external forces and general boundary conditions.

In a recent paper, Marshall and Watson (1987) obtained an analytic solution of the albedo and Milne problems for the stationary one-dimensional Fokker-Planck equation with constant force term. The problem is of interest for chemical kinetics, coagulation studies and, in general, for the computation of first-passage times in any Brownian (Ornstein-Uhlenbeck) process. Apparently, it was first considered by Wang and Uhlenbeck (1945) who proposed, however, only heuristic methods and approximate solutions. More elaborate and accurate numerical calculations have been recently performed by Burschka and Titulaer (1982), Mayya and Sahni (1983) and Selinger and Titulaer (1984). An extensive and virtuoso analytic study of the Fokker-Planck equation by using methods from the theory of special functions has been carried out by Pagani (1970). Recently, Marshall and Watson (1985) and Dita (1985) obtained explicit solutions to the albedo and Milne problems for the Fokker-Planck equation by exploiting specific properties of the Weber functions associated with the respective problems. Finally, an approach to solving the stationary Fokker-Planck equation, based on using the Green function method for partial differential equations, has been furthered by Chandrasekhar (1943), Weber (1951), Menon *et al* (1986), Marshall and Watson (1987) and Dita (1987).

Almost a decade ago, it became clear that the stationary one-dimensional Fokker-Planck equation falls into the category of indefinite (backward-forward) Sturm-Liouville transport problems and that special methods of functional analysis and operator theory as outlined by Baouendi and Grisvard (1968) and Beals (1977, 1979, 1981) may prove more appropriate and more powerful for an abstract treatment. Following this programme, half-range completeness results have been proved by Beals and Protopopescu (1983, 1984) for the Fokker-Planck equation itself and by Beals (1985) for general indefinite Sturm-Liouville operators. Constructive representations of solutions by two equivalent methods, namely, direct eigenfunction expansion and Wiener-Hopf factorisation, have been obtained by Klaus *et al* (1985, 1987). A state of the art of the abstract theory for indefinite Sturm-Liouville transport problems, together with several typical examples, can be found in the recent monograph by Greenberg *et al* (1987, ch 10).

The analytic solutions of the albedo and Milne problem for the Fokker-Planck equation with constant external force, reported by Marshall and Watson (1985, 1987), were found under the assumption that the half-range completeness established by Beals and Protopopescu (1983, 1984) in the force-free case is valid also for a non-zero external force. Accordingly, the solution is developed in an infinite set of independent vectors, yet the coefficients of the development contain themselves infinite sums that are difficult to compute and eventually are amenable only to approximate results.

Relying on several observations, we present here a generalisation and simplification of the results of Marhsall and Watson (1985, 1987) in the following sense.

(i) We shall show that the assumed half-range completeness is indeed valid in the presence of an external force that, moreover, does not need to be restricted to a constant.

(ii) We shall include not only the Milne and albedo problems, but a wide range of transport problems involving general partially reflecting and partially absorbing boundary conditions.

(iii) We shall derive a half-range orthogonality relation and we shall apply it to the Fokker-Planck problem with constant external force. This will be done by using a special decomposition of the original indefinite Sturm-Liouville problem into a couple of definite Sturm-Liouville problems.

Specifically, we shall consider the stationary one-dimensional Fokker-Planck equation with force term

$$v\frac{\partial\Psi}{\partial x}=\frac{\partial^2\Psi}{\partial v^2}+(v+2\alpha(v))\frac{\partial\Psi}{\partial v}+\Psi \tag{1}$$

for the one-particle distribution function $\Psi(x, v)$ depending on the position x, $x>0$, and on the velocity v, $v\in(-\infty,\infty)$. Here, the (non-constant) external force per unit mass is $2\alpha(v)$. From the physical viewpoint, much greater interest attaches to the case of a force *field*, for which α is (also) a function of x (see, for instance, Duck *et al* (1986)). However, when the stationary equation (1) is considered, as in the following, as a backward-forward evolution problem with x playing the role of evolution parameter, only forces depending on v alone can be included in the treatment. We shall specify the boundary conditions (BC) at $x=0$ and $x\to\infty$ as is usually done in kinetic or transport theory, namely:

$$\Psi(0, v)=\Phi(v)+(RJ\Psi)(0, v) \qquad v>0 \tag{2}$$

$$\Psi(x, v)\approx o(1) \qquad \text{or} \qquad O(x^n) \qquad n=0,1,\dots \qquad \text{for } x\to\infty. \tag{3}$$

In BC (3), that specifies the distribution at $x=0$ only for a half-range of velocities, J is the inversion operator, $J\Psi(v)=\Psi(-v)$, and R is the boundary operator. The action of R depends on the surface scattering law, e.g., $R=0$ for a purely absorbing boundary, $R=I$ (the identity operator), for a specularly reflecting boundary, etc. In general, the BC at infinity may or may not be compatible with the rest of the problem. Accordingly, the problem may sometimes have no solution while sometimes it may have multiple solutions. For a detailed discussion of this point we refer the interested reader to Greenberg *et al* (1987, ch 3, 4). In the following, we shall take BC (3) in the form $\Psi(x, v)\approx c$ for $x\to\infty$, where the constant c has to be determined. Other growth conditions at infinity can be treated likewise.

First, we want to show that the half-range completeness result covers the problem (1)-(3) with rather general $\alpha(v)$ and $R=0$.

To this aim, we shall perform the transformation

$$\Psi(x, v)=\exp\left(-\tfrac{1}{2}v^2-2\int^v\alpha(u)\,\mathrm{d}u\right)\psi(x, v). \tag{4}$$

Inserting (4) into (1)-(3) and separating the variables in the form

$$\psi(x, v) = e^{-\lambda x}\psi(v) \tag{5}$$

we can cast (1)-(3) in the standard indefinite Sturm-Liouville form

$$-\frac{d}{dv}\left(p(v)\frac{d\psi(v)}{dv}\right) + q(v)\psi(v) = \lambda w(v)\psi(v) \tag{6}$$

$$\psi(0, v) = \phi(v) + (RJ\psi)(0, v) \qquad v > 0 \tag{7}$$

$$\psi(x, v) = c \qquad \text{for } x \to \infty \tag{8}$$

where

$$w(v) = v \exp\left(-\tfrac{1}{2}v^2 - 2\int^v \alpha(u)\, du\right) \tag{9}$$

$$p(v) = \exp\left(-\tfrac{1}{2}v^2 - 2\int^v \alpha(u)\, du\right) \tag{10}$$

$$q(v) = 2\alpha'(v) \exp\left(-\tfrac{1}{2}v^2 - 2\int^v \alpha(u)\, du\right) \tag{11}$$

$$\phi(v) = \Phi(v) \exp\left(-\tfrac{1}{2}v^2 - 2\int^v \alpha(u)\, du\right). \tag{12}$$

As in the original formulation (1)-(3), the important feature of the problem is that the function $w(v)$ changes sign on $(-\infty, \infty)$, namely at $v = 0$. One sign change is usually the most common situation, but several sign changes can be treated likewise (Greenberg *et al* 1987, ch 10).

The transformation (4) imposes some requirements on the external force: $\alpha(v)$ must be locally continuously differentiable such that $q(v)$ be continuous on R. If $\alpha(v) < 0$, an extra growth condition has to be imposed for large v, namely, there exists $\delta > 0$, $C > 0$ such that

$$|\alpha(v)| \leq C|v|^{1-\delta} \qquad |v| \to \infty. \tag{13}$$

This condition ensures that $w(v)$, $p(v)$, $q(v)$ remain bounded for all v. Moreover, $\alpha(v)$ must be such that the spectrum of the operator $-(d/dv)p(v)\, d/dv + q(v)$ be contained in $\{0\} \cup [\varepsilon, \infty)$; $\varepsilon > 0$.

Once the problem is set up in the standard form (6), its solvability as well as the half-range completeness property for $R = 0$ follows directly from Greenberg *et al* (1987, ch 10, theorem 1.5, lemma 2.1 and corollary 2.2).

The extension to $R \neq 0$, $R \leq 1$ is achieved via a fixed point argument, in the same spirit as that in which it was applied to time-dependent transport theory by Beals and Protopopescu (1987). Let us write the solution at $x = 0$ of the problem (6)-(8) with $R = 0$ in the usual notation:

$$\psi(0, v) = (E\phi)(0, v) \tag{14}$$

where $\phi(v)$, $v > 0$, is the given surface term wherefrom the complete distribution at $x = 0$, $\psi(0, v)$, all v, is uniquely constructed by the action of the albedo operator E. We recall that the complete knowledge of ψ at $x = 0$ enables one to construct immediately ψ at any $x > 0$. For the general problem (6)-(8), with $R \neq 0$, we look for a solution

$$\psi(0, v) = (E\phi^*)(0, v) \tag{15}$$

where $\phi^*(v)$, $v>0$, is the new surface source term to be determined uniquely. Like $\phi(v)$, $\phi^*(v)$ is defined only for $v>0$ and, when considered as a function of R, will be taken identically zero for $v<0$.

Inserting (15) into (7) and taking into account that $E\phi$ solves (7) with $R=0$, we obtain

$$\psi(0, v>0)=\phi^*(v)=(RJE\phi^*)(0, v>0)+\phi(v). \tag{16}$$

Taking into account that

$$(E\phi^*)(v<0)=(E\phi^*)(v)-\phi^*(v)=(E-1)\phi^*(v) \tag{17}$$

we obtain finally

$$\phi^*=RJ(E-1)\phi^*+\phi \tag{18}$$

and

$$\phi^*=[1+RJ(1-E)]^{-1}\phi. \tag{19}$$

Since $\|1-E\|<1$ and J is an involution ($\|J\|=1$), the invertibility of $[1+RJ(1-E)]$ is ensured for any boundary reflection operators with $\|R\|\leqslant 1$. This proves the solvability of the stationary problem (6)-(8) for any $\alpha(v)$ and R satisfying the conditions discussed above and, implicitly, the existence of a half-range completeness result. In general, the half-range complete functions are not known special functions; however, for $\alpha(v)=\alpha=$ constant, these functions are the Weber functions (see, e.g., Olver 1974).

The set of orthonormal half-range complete Weber functions is specified by BC for both x and v as will be seen in the following. The idea is to reduce the *indefinite* eigenvalue problem, as considered on $(-\infty, \infty)$, to a pair of two *definite* eigenvalue problems considered on $(-\infty, 0)$ and $(0, \infty)$, respectively. In order to achieve this decomposition, one has to add a supplementary BC at $v=0$. Since it is not imposed by the physical requirements of the original problem, this supplementary BC is, to a large extent, arbitrary and will be chosen such that the two reduced Sturm-Liouville operators obtained from the original operator be positive and self-adjoint in $L_2((-\infty, 0); |w(v)|\,dv)$ and $L_2((0; \infty); w(v)\,dv)$, respectively. Thus, they possess complete *orthonormal* systems of eigenfunctions in the respective spaces (Morse and Feshbach 1953).

We shall achieve the reduction of the indefinite Sturm-Liouville problem (6)-(8) by imposing at $v=0$ a separated BC of Neumann-Dirichlet type:

$$\psi(0)\cos\beta-p(0)\psi'(0)\sin\beta=0 \qquad \beta\in[0, \pi). \tag{20}$$

This type of reduction has been previously carried out and applied by Klaus *et al* (1985, 1987) to the Wiener-Hopf factorisation of general Sturm-Liouville transport problems with any finite number of sign changes. This reduction also appears, although not in explicit terms, in Marshall and Watson (1985, appendix 1), who even wrote a half-range orthogonality relation corresponding to the case $\beta=\pi/2$ in (20). They did not, however, use it in their work so they only obtained the solution in the form of infinite sums as given by the traditional, but less convenient, set $\{D_n(v);\ n=0, 1, 2, \ldots\}$ which is half-range complete, but not half-range orthogonal. Dita (1985, 1986) used the decomposition independently and realised that one can thus obtain the explicit solution of the albedo and Milne problems for the force-free Fokker-Planck equation via expansions in eigenfunctions that are half-range complete *and* half-range orthogonal. Dita took $\beta=0$ in (20). The existence of a half-range orthogonality relation is important for concrete applications seeking efficient and accurate numerical results, since it yields the coefficients of the eigenfunction expansion via simple quadratures. We shall include here the calculation for the Fokker-Planck equation with constant force as another application of interest. We stress that the method is general, but its usefulness for applications depends on the explicit knowledge of the orthonormal complete set, which is not guaranteed for more general forces $\alpha(v)$.

For the Fokker-Planck equation with $\alpha(v)=\alpha=$ constant, the separation of variables of the form

$$\psi(x, v)=\mathrm{e}^{-\lambda x}\,\mathrm{e}^{-(v^2/4+\alpha v)}\phi(v) \tag{21}$$

leads to the equation

$$\phi''(v)+(\tfrac{1}{2}-\alpha^2+v(\lambda-\alpha)-\tfrac{1}{4}v^2)\phi(v)=0 \tag{22}$$

which is satisfied by the Weber function $D_{\lambda^2-2\alpha\lambda}(v-2(\lambda-\alpha))$ (Morse and Feshbach 1953). This function tends to zero when v tends to infinity; the other independent solution, $D_{-\lambda^2+2\alpha\lambda-1}(\mathrm{i}[v-2(\lambda-\alpha)])$, is not acceptable because of its oscillatory behaviour at infinity. The set of eigenvalues and the corresponding set of complete orthonormal eigenfunctions are determined by imposing the BC (20) at $v=0$.

For instance, if we take $\beta=0$, then the BC (20) applied to the acceptable solution of (22) are

$$D_{\lambda^2-2\alpha\lambda}(-2(\lambda-\alpha))=0. \tag{23}$$

The solutions of (23) determine an infinite denumerable set of real distinct eigenvalues λ_n for the original Sturm-Liouville problem. The corresponding eigenfunctions, $\{\phi_n(v;\lambda_n)\}$ where $\phi_n(v;\lambda_n)\equiv D_{\lambda_n^2-2\alpha\lambda_n}(v-2(\lambda_n-\alpha))$ satisfy the usual orthogonality relation satisfied by a Sturm-Liouville set:

$$\int_0^\infty w(v)\phi_m(v)\phi_n(v)\,\mathrm{d}v=\int_0^\infty v\phi_m(v)\phi_n(v)\,\mathrm{d}v=\delta_{nm}. \tag{24}$$

If we compare the set $\{\phi_n(v;\lambda_n)\}$ with the set $\{D_n(2(n-\alpha^2)^{1/2}-v)\}$, $n\in Z_+$, used by Marshall and Watson (1987), we see that the eigenvalues λ_n are more difficult to compute, but the eigenfunctions $\phi_n(v;\lambda_n)=D_{\lambda_n^2-2\alpha\lambda_n}(v-2(\lambda_n-\alpha))$ satisfy simpler relations than do the eigenfunctions $D_n(2(n-\alpha^2)^{1/2}-v)$, namely the half-range orthogonality relation (24) that yields readily the expansion coefficients. We note that the way to achieve orthogonality by passing from $\{n\in Z_+;\ D_n(2(n-\alpha^2)^{1/2}-v)\}$ to $\{\lambda_n;\ D_{\lambda_n^2-2\alpha\lambda_n}(v-2(\lambda_n-\alpha))\}$ differs from the classical method of the Chandrasekhar weight function used in neutron transport and gas dynamics. We note also that the possibility of the half-range orthogonality being realised by such an operatorial transformation aimed at changing the basis, and not by a simple multiplicative weight function applied to the old (full-range) basis, had been considered by Beals and Protopopescu (1984), but no definitive solution was found at that time.

In conclusion, we have shown in this letter how to generalise some recent results of Marshall and Watson (1987) concerning the stationary problem for the one-dimensional Fokker-Planck equation with external force term. Relying on the results of the abstract theory for kinetic equations, we were able to include more general force terms and boundary conditions. Moreover, by transforming the indefinite Sturm-Liouville problem into a pair of definite ones, we could construct a set of eigenfunctions that are both half-range complete and half-range orthogonal and we exemplified the construction for the constant-force case.

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